DAWSON COLLEGE
Mathematics Department
Final Examination
Linear Algebra
201-NYC-05 Sections 01, 02, 03, 04, 05, 06
December 19th, 2017
9:30-12:30

Student Name ___Solutions___

Student I.D. # _____

Teacher _____

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Instructions:
- Print your name and student ID number in the space provided above.
- All questions are to be answered directly on the examination paper in the provided space.
- Translation and regular dictionaries are permitted.
- The Sharp EL-531XG or EL-531X calculators are permitted.
- This examination consists of 17 questions.
- This exam booklet must be returned intact.

Question #	Mark
1	/5
2	/5
3	/5
4	/4
5	/6
6	/4
7	/8
8	/8
9	/4
10	/8
11	/4
12	/5
13	/12
14	/8
15	/4
16	/6
17	/4
Total	/100

1. [4+1 marks]

a) Determine the general solution of the following system using Gauss-Jordan Elimination.

$$\begin{cases} x_1 \quad\quad + 2x_3 + x_4 = -1 \\ 2x_1 + x_2 + 3x_3 + 3x_4 = 2 \\ -x_1 + x_2 - 3x_3 \quad\quad = 5 \\ 3x_1 + 2x_2 + 7x_3 + 2x_4 = -4 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & -1 \\ 2 & 1 & 3 & 3 & 2 \\ -1 & 1 & -3 & 0 & 5 \\ 3 & 2 & 7 & 2 & -4 \end{array}\right] \begin{array}{l} r_2 : r_2 - 2R_1 \\ \xrightarrow{} \\ r_3 : r_3 + r_1 \\ r_4 : r_4 - 3r_1 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & 1 & 4 \\ 0 & 1 & -1 & 1 & 4 \\ 0 & 2 & 1 & -1 & -1 \end{array}\right] \begin{array}{l} r_3 : r_3 - r_2 \\ \xrightarrow{} \\ r_4 : r_4 - 2r_2 \\ (r_3 \leftrightarrow r_4) \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & 1 & 4 \\ 0 & 0 & 3 & -3 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_3 : \frac{1}{3}r_3} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & 1 & 4 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} r_1 : r_1 - 2r_3 \\ \xrightarrow{} \\ r_2 : r_2 + r_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 5 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\begin{array}{l} X_1 = 5 - 3t \\ X_2 = 1 \\ X_3 = -3 + t \\ X_4 = t \end{array} \qquad t \in \mathbb{R}}$$

b) Find the particular solution when $x_1 = 2$

$$2 = 5 - 3t$$
$$-3 = -3t$$
$$\Rightarrow \boxed{t = 1}$$

$$\boxed{\begin{array}{l} X_1 = 2 \\ X_2 = 1 \\ X_3 = -2 \\ X_4 = 1 \end{array}}$$

2. *[5 marks]* Determine the conditions on a such that the system has

a) no solution   b) one solution   c) infinitely many solutions.

$$x + 4y - 10z = 1$$
$$-3x + 2y + 2z = 2$$
$$4x + 2y + (a^2 - 13)z = a$$

$$\begin{bmatrix} 1 & 4 & -10 & \bigm| & 1 \\ -3 & 2 & 2 & \bigm| & 2 \\ 4 & 2 & a^2-13 & \bigm| & a \end{bmatrix} \xrightarrow[r_3 : r_3 - 4r_1]{r_2 : r_2 + 3r_1} \begin{bmatrix} 1 & 4 & -10 & \bigm| & 1 \\ 0 & 14 & -28 & \bigm| & 5 \\ 0 & -14 & a^2+27 & \bigm| & a-4 \end{bmatrix} \xrightarrow{r_3 : r_3 + r_2}$$

$$\begin{bmatrix} 1 & 4 & -10 & \bigm| & 1 \\ 0 & 14 & -28 & \bigm| & 5 \\ 0 & 0 & a^2-1 & \bigm| & a+1 \end{bmatrix}$$

a) no sol'n : $a = 1$

b) one sol'n : $a \neq \pm 1$, $a \in \mathbb{R}$

c) infinitely many sol'ns : $a = -1$

3. *[3+2 marks]* Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 2 & 4 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Evaluate (wherever possible):

a) $A^{-1}BC = -1 \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}_{2\times2} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 2 & 4 & -1 \end{bmatrix}_{2\times4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{4\times4}$

$= -1 \begin{bmatrix} 5 & -14 & -3 & 17 \\ -3 & 8 & 1 & -10 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

$= -1 \begin{bmatrix} 5 & -28 & -6 & 51 \\ -3 & 16 & 2 & -30 \end{bmatrix}$

$= \begin{bmatrix} -5 & 28 & 6 & -51 \\ 3 & -16 & -2 & 30 \end{bmatrix}$

b) C^{-4}

$\begin{bmatrix} 1^4 & 0 & 0 & 0 \\ 0 & (1/2)^4 & 0 & 0 \\ 0 & 0 & (1/2)^4 & 0 \\ 0 & 0 & 0 & (1/3)^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/16 & 0 & 0 \\ 0 & 0 & 1/16 & 0 \\ 0 & 0 & 0 & 1/81 \end{bmatrix}$

4. *[4 marks]* Solve for X when:

$$\begin{bmatrix} -13 & 1 \\ 27 & 2 \end{bmatrix} X = \begin{bmatrix} -11 & \frac{1}{2} \\ 29 & 1 \end{bmatrix} X - \begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix}$$

$\left(\begin{bmatrix} -13 & 1 \\ 27 & 2 \end{bmatrix} - \begin{bmatrix} -11 & 1/2 \\ 29 & 1 \end{bmatrix} \right) X = - \begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix}$

$\begin{bmatrix} -2 & 1/2 \\ -2 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}$

$X = \begin{bmatrix} -2 & 1/2 \\ -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}$

$= - \begin{bmatrix} 1 & -1/2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix} = - \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -6 & -4 \end{bmatrix}$

5. [3+3 marks]

a) Let $A = \begin{bmatrix} 1 & 2 & 0 \\ -3 & -5 & 0 \\ 2 & 4 & 1 \end{bmatrix}$; express A^{-1} as a product of elementary matrices

$$\begin{bmatrix} 1 & 2 & 0 \\ -3 & -5 & 0 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow{r_2 : r_2 + 3r_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow{r_3 : r_3 - 2r_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 : r_1 - 2r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 E_2 E_1 A = I \quad \therefore \quad A^{-1} = E_3 E_2 E_1 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) Prove that if A is an invertible matrix and B has the same reduced row echelon form as A, then B is also invertible

A is invertible $\Rightarrow$ I is the reduced row echelon form of A

if B has the same reduced echelon form as A
$\Rightarrow$ I is the reduced row echelon form of B
$\Rightarrow$ B is invertible

6. *[4 marks]* Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & k & 1 \\ 2 & 4 & k \end{bmatrix}$

Determine the values of k for which the homogeneous system $Ax = 0$ has only the trivial solution

trivial sol'n whenever $\begin{vmatrix} 1 & 2 & 0 \\ 1 & K & 1 \\ 2 & 4 & K \end{vmatrix} = 1(k^2-4) - 2(k-2) \neq 0$

$k^2 - 4 - 2k + 4 \neq 0$

$K(K-2) \neq 0$

trivial sol'n : $\boxed{K \neq 0, \ K \neq 2, \ K \in \mathbb{R}}$
whenever

7. *[4+4 marks]* Let A, B, C be 4x4 matrices such that $det(A) = 2; \quad det(B) = 3; \quad det(C) = -1$
Evaluate the following:

a) $det(2A^T(3B)^{-1}C^{22}) = det(2A^T) \, det[(3B)^{-1}] \, det(C^{22})$

$= 2^4 det(A) \cdot \dfrac{1}{det(3B)} \cdot [det(C)]^{22}$

$= 2^4 \cdot 2 \cdot \dfrac{1}{3^4 det(B)} \cdot (-1)^{22}$

$= \dfrac{2^5}{3^4 \cdot 3} = \dfrac{2^5}{3^5} = \boxed{\dfrac{32}{243}}$

b) $det(AB^{-1} - A\,adj(B))$

$= det(A(B^{-1} - adj(B))) = det(A)\,det(B^{-1} - adj(B))$

$= 2\, det(B^{-1} - det(B)\cdot B^{-1})$

$= 2\, det(B^{-1} - 3B^{-1})$

$= 2\, det(-2B^{-1})$

$= 2(-2)^4 det(B^{-1})$

$= \dfrac{2^5}{det(B)} = \boxed{\dfrac{32}{3}}$

$$\begin{cases} x + 2y = 1 \\ x + 3y + z = -2 \\ 2x + 4y + 3z = 0 \end{cases}$$

a) Find A^{-1} using $adj(A)$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 2 & 4 & 3 \end{bmatrix} \qquad adj(A) = \begin{bmatrix} 5 & -1 & -2 \\ -6 & 3 & 0 \\ 2 & -1 & 1 \end{bmatrix}^T \; ; \qquad det(A) = 3$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 5 & -6 & 2 \\ -1 & 3 & -1 \\ -2 & 0 & 1 \end{bmatrix}$$

b) Use your answer from part a) to determine the solution to the system

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 5 & -6 & 2 \\ -1 & 3 & -1 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/3 \\ -5/3 \\ 2/3 \end{bmatrix}$$

Cramer's rule is not used):

$$x - y + 4z = 10$$
$$-2x + y + z = 0$$
$$4x - y + 2z = 6$$

$$\det(A) = \begin{vmatrix} 1 & -1 & 4 \\ -2 & 1 & 1 \\ 4 & -1 & 2 \end{vmatrix} = 4(-2) - 1(3) + 2(-1) = -13$$

$$\det(A_z) = \begin{vmatrix} 1 & -1 & 10 \\ -2 & 1 & 0 \\ 4 & -1 & 6 \end{vmatrix} = 10(-2) - 0(3) + 6(-1) = -26$$

$$z = \frac{\det(A_z)}{\det(A)} = \frac{-26}{-13} = \boxed{2}$$

a) the area of the triangle determined by the two vectors $(\vec{AB} \times \vec{AC})$ and $\vec{BC}$

$$\vec{AB} = (2, -1, 1), \quad \vec{AC} = (-1, 2, 0), \quad \vec{BC} = (-3, 3, -1)$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ -1 & 2 & 0 \end{vmatrix} = (-2, -1, 3)$$

$$(\vec{AB} \times \vec{AC}) \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -1 & 3 \\ -3 & 3 & -1 \end{vmatrix} = (-8, -11, -9)$$

$$\| (\vec{AB} \times \vec{AC}) \times \vec{BC} \| = \sqrt{64 + 121 + 81} = \sqrt{266}$$

$$\boxed{Area_\Delta = \frac{\sqrt{266}}{2} \ units}$$

a) the angle between $(\vec{AB} \times \vec{AC})$ and $\vec{BC}$

$$angle = \cos^{-1}\left(\frac{(\vec{AB} \times \vec{AC}) \cdot (\vec{BC})}{\| (\vec{AB} \times \vec{AC}) \| \ \| \vec{BC} \|} \right) = \cos^{-1}\left(\frac{(-2, -1, 3) \cdot (-3, 3, -1)}{\sqrt{14} \cdot \sqrt{19}} \right)$$

$$= \cos^{-1}\left(\frac{0}{\sqrt{14}\sqrt{19}} \right) = \cos^{-1}(0) = \boxed{\pi \ radians \ or \ 90°}$$

or
$\vec{AB} \times \vec{AC}$ is a normal to the plane containing points A, B & C ∴ orthogonal to $\vec{BC}$

Let $\vec{s} = (\vec{v} - \vec{w})$ and $\vec{t} = (\vec{u} \times \vec{v}) + (\vec{v} \times \vec{w}) + (\vec{w} \times \vec{u})$. Show that $\vec{s}$ and $\vec{t}$ are orthogonal

$\vec{s} \cdot \vec{t} =$

$(\vec{v} - \vec{w}) \cdot \left[(\vec{u} \times \vec{v}) + (\vec{v} \times \vec{w}) + (\vec{w} \times \vec{u}) \right]$

$= \vec{v} \cdot (\vec{u} \times \vec{v}) + \vec{v} \cdot (\vec{v} \times \vec{w}) + \vec{v} \cdot (\vec{w} \times \vec{u})$

$\quad - \left[\vec{w} \cdot (\vec{u} \times \vec{v}) + \vec{w} \cdot (\vec{v} \times \vec{w}) + \vec{w} \cdot (\vec{w} \times \vec{u}) \right]$

$= 0 + 0 + \vec{v} \cdot (\vec{w} \times \vec{u}) - \left[\vec{w} \cdot (\vec{u} \times \vec{v}) + 0 + 0 \right]$

$= \vec{v} \cdot (\vec{w} \times \vec{u}) - \vec{w} \cdot (\vec{u} \times \vec{v})$

$= \vec{v} \cdot (\vec{w} \times \vec{u}) + \vec{w} \cdot (\vec{v} \times \vec{u})$

$= \vec{v} \cdot (\vec{w} \times \vec{u}) - \vec{v} \cdot (\vec{w} \times \vec{u})$

$= 0$

$\therefore$ vectors are orthogonal

12. *[3+2 marks]* Given the plane $-2x + 3y = -z - 17$, the point $A(4, -2, 2)$, and the line

$$\begin{cases} x = -1 - 2t \\ y = 4 - t \quad t \in \mathbb{R} \\ z = -5 + 6t \end{cases}$$

a) Find the point of intersection between the line and the plane

$-2(-1-2t) + 3(4-t) + (-5+6t) = -17$

$2 + 4t + 12 - 3t - 5 + 6t = -17$

$7t = -26 \qquad t = -\dfrac{26}{7}$

$x = -1 + \dfrac{52}{7} = \dfrac{45}{7}$

$y = 4 + \dfrac{26}{7} = \dfrac{54}{7}$

$\boxed{\left(\dfrac{45}{7}, \dfrac{54}{7}, -\dfrac{191}{7} \right)}$

$z = -5 - \dfrac{156}{7} = -\dfrac{191}{7}$

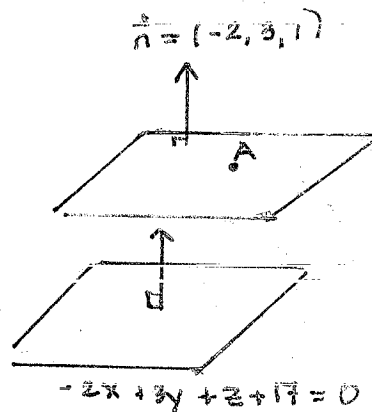b) Find the general equation of the plane containing the point A and parallel the given plane

$\vec{n} = (-2, 3, 1) \qquad A(4, -2, 2)$

eq'n:

$-2(x-4) + 3(y+2) + (z-2) = 0$

$-2x + 8 + 3y + 6 + z - 2 = 0$

$\boxed{-2x + 3y + z + 12 = 0}$

$\vec{n} = (-2, 3, 1)$

A

d

$-2x + 3y + z + 17 = 0$

13. *[4+4+4 marks]* Given the point $P(2, -6, 8)$ and the lines:

$$l_1: \begin{cases} x = -2 + 3t \\ y = 1 \\ z = 3t \end{cases} \quad \text{and} \quad l_2: \begin{cases} x = s \\ y = -3s \\ z = 3 - 2s \end{cases} \quad s, t \in \mathbb{R}$$

l_1: thru $R(-2, 1, 0)$
w/ directional vector
$\vec{v_1} = (3, 0, 3)$

l_2: thru $Q(0, 0, 3)$
$\vec{v_2} = (1, -3, -2)$

a) compute the shortest distance between them

$\begin{aligned} s - 3t &= -2 \\ -3s &= 1 \\ -2s - 3t &= -3 \end{aligned}$

$\begin{bmatrix} 1 & -3 & | & -2 \\ -3 & 0 & | & 1 \\ -2 & -3 & | & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & | & -2 \\ 0 & -9 & | & -5 \\ 0 & -9 & | & -7 \end{bmatrix}$ no sol'n

$\therefore l_1 \& l_2$ do not intersect; $\because \vec{v_1} \neq k\vec{v_2}$
$\therefore$ lines not parallel
$\Rightarrow l_1 \& l_2$ SKEW LINES

$\vec{RQ} = (2, -1, 3)$ $\vec{v_1} \times \vec{v_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & 3 \\ 1 & -3 & -2 \end{vmatrix} = (9, 9, -9)$
$= 9(1, 1, -1)$

let $\vec{n} = (1, 1, -1)$

$d = \| proj_{\vec{n}} \vec{RQ} \| = \dfrac{|\vec{n} \cdot \vec{RQ}|}{\|\vec{n}\|} = \dfrac{|(1,1,-1) \cdot (2,-1,3)|}{\sqrt{3}} = \dfrac{|-2|}{\sqrt{3}} = \boxed{\dfrac{2}{\sqrt{3}} \text{ units}}$

b) find the point on l_1 that is closest to the point P

let A be the pt on l_1 closest to P: $A(-2 + 3t, 1, 3t)$

$\vec{PA} = (-2 + 3t - 2, 1 - (-6), 3t - 8)$
$= (-4 + 3t, 7, -8 + 3t)$

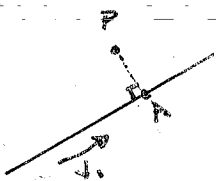$\vec{PA} \cdot \vec{v_1} = 0: \quad (-4 + 3t, 7, -8 + 3t) \cdot (3, 0, 3) = 0$
$= -12 + 9t + 0 - 24 + 9t = 0$
$18t = 36$
$t = 2$

$A(-2 + 3(2), 1, 3(2))$
$\boxed{A(4, 1, 6)}$

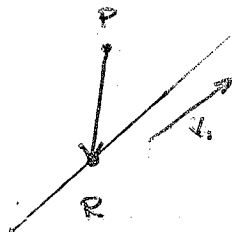c) determine the general equation of the plane containing the line l_1 and the point P

$\vec{v_1} = (3, 0, 3)$ $\vec{PR} = (-2 - 2, 1 - (-6), 0 - 8)$
$= (-4, 7, -8)$

$\vec{v_1} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & 3 \\ -4 & 7 & -8 \end{vmatrix} = (-21, 12, 21)$
$= 3(-7, 4, 7)$

let $\vec{n} = (-7, 4, 7)$

using P:
$-7(x - 2) + 4(y + 6) + 7(z - 8) = 0$
$-7x + 14 + 4y + 24 + 7z - 56 = 0$
$\boxed{-7x + 4y + 7z - 18 = 0}$

14. *[4 +4 marks]* Let V be the set of 2x2 skew symmetric matrices, i.e. $\mathcal{V} = \{A|\ A^T = -A\}$, with the usual operations of addition and scalar multiplication on matrices.

a) Show that $\mathcal{V}$ a subspace of M_{22} $\quad \mathcal{V}$ not empty $\quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathcal{V}$

A1: let $B \in \mathcal{V}$, $(A+B)^T = A^T + B^T$
$$= -A + (-B)$$
$$= -(A+B) \quad \therefore \ \mathcal{V} \text{ closed under}$$
$$\text{vector addition}$$

S1: (or A6)
let $K \in \mathbb{R}$, $(KA)^T = K A^T = K(-A) = -K(A) = -(KA)$
$$\therefore \ \mathcal{V} \text{ closed under}$$
$$\text{scalar multiplication}$$

$\therefore \ \mathcal{V}$ is a subspace
of M_{22}

b) Determine a basis and calculate the dimension of the subspace $\mathcal{V}$

$A \in \mathcal{V} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$

$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \Rightarrow \begin{matrix} a = d = 0 \\ b = -c \end{matrix}$

$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$

$\boxed{\text{basis}_{\mathcal{B}} = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}}$ $\qquad \boxed{\dim(\mathcal{V}) = 1}$

15. *[4 marks]* Let $\vec{u}, \vec{v},$ and $\vec{w}$ denote vectors in a vector space $\mathcal{V}$. Show that

$$span\{\vec{u}, \vec{v}, \vec{w}\} \subseteq span\{\vec{u} - \vec{v}, \vec{u} + \vec{w}, \vec{w}\}$$

need to show $\exists\ r, s, t \ni$

i) $\vec{u} = r(\vec{u} - \vec{v}) + s(\vec{u} + \vec{w}) + t(\vec{w}) \Rightarrow r = 0,\ s = 1,\ t = -1$

ii) $\vec{v} = r(\vec{u} - \vec{v}) + s(\vec{u} + \vec{w}) + t(\vec{w}) \Rightarrow r = -1,\ s = 1,\ t = -1$

iii) $\vec{w} = r(\vec{u} - \vec{v}) + s(\vec{u} + \vec{w}) + t(\vec{w}) \Rightarrow r = 0,\ s = 0,\ t = 1$

$\therefore$ span $\{\vec{u}, \vec{v}, \vec{w}\} \subseteq$ span $\{\vec{u} - \vec{v},\ \vec{u} + \vec{w},\ \vec{w}\}$

16. *[3 +3 marks]* Let A be the coefficient matrix of a homogeneous system.

$$A = \begin{bmatrix} 1 & 6 & 4 & 1 & 4 & 3 \\ 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 3 & 0 & 0 & 0 \end{bmatrix}$$

a) Find the solution to the system.

$r_1 ; r_1 - 6r_2$
$\rightarrow$
$\begin{bmatrix} 1 & 0 & -2 & -5 & 4 & -15 \\ 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 3 & 0 & 0 & 0 \end{bmatrix}$
$r_1 ; r_1 - 4r_3$
$\longrightarrow$
$\overset{r \quad\quad s \quad\quad\quad t}{\begin{bmatrix} 1 & 0 & -2 & -5 & 0 & -35 \\ 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 3 & 0 & 0 & 0 \end{bmatrix}}$

$X_1 = 2r + 5s + 35t$
$X_2 = -1r - 1s - 3t$
$X_3 = r$
$X_4 = s$
$X_5 = -5t$
$X_6 = t$

Vector eq'n.

$\mathbf{X} = r(2,-1,1,0,0,0) + s(5,-1,0,1,0,0) + t(35,-3,0,0,-5,1)$

b) Find a basis and the dimension for solution space

$B = \{ (2,-1,1,0,0,0), (5,-1,0,1,0,0), (35,-3,0,0,-5,1) \}$

dimension = 3

$$\{1 + x, 3x + x^2, 2 + x - kx^2\}$$

$P_2: a_0 + a_1 x + a_2 x^2 \qquad a_0, a_1, a_2 \in \mathbb{R}$

$r(1 + x) + s(3x + x^2) + t(2 + x - kx^2) = 0 + 0x + 0x^2$

$r + 0s + 2t = 0 \qquad$ (coefficients of x^0)
$r + 3s + t = 0 \qquad$ (coefficients of x^1)
$0r + s - k = 0 \qquad$ (coefficients of x^2)

trivial sol'n (hence linearly independent in P_2)
as long as $\det(A) \neq 0$

where $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 1 \\ 0 & 1 & -k \end{bmatrix}$

$\det(A) = 1(-3k - 1) + 2(1)$

$= -3k + 2$

$\neq 0$ if $\boxed{k \neq \dfrac{3}{2}, \ k \in \mathbb{R}}$